FOR THE
IB DIPLOMA
PROGRAMME

ANSWERS

Mathematics

ANALYSIS AND APPROACHES SL

EXAM PRACTICE WORKBOOK

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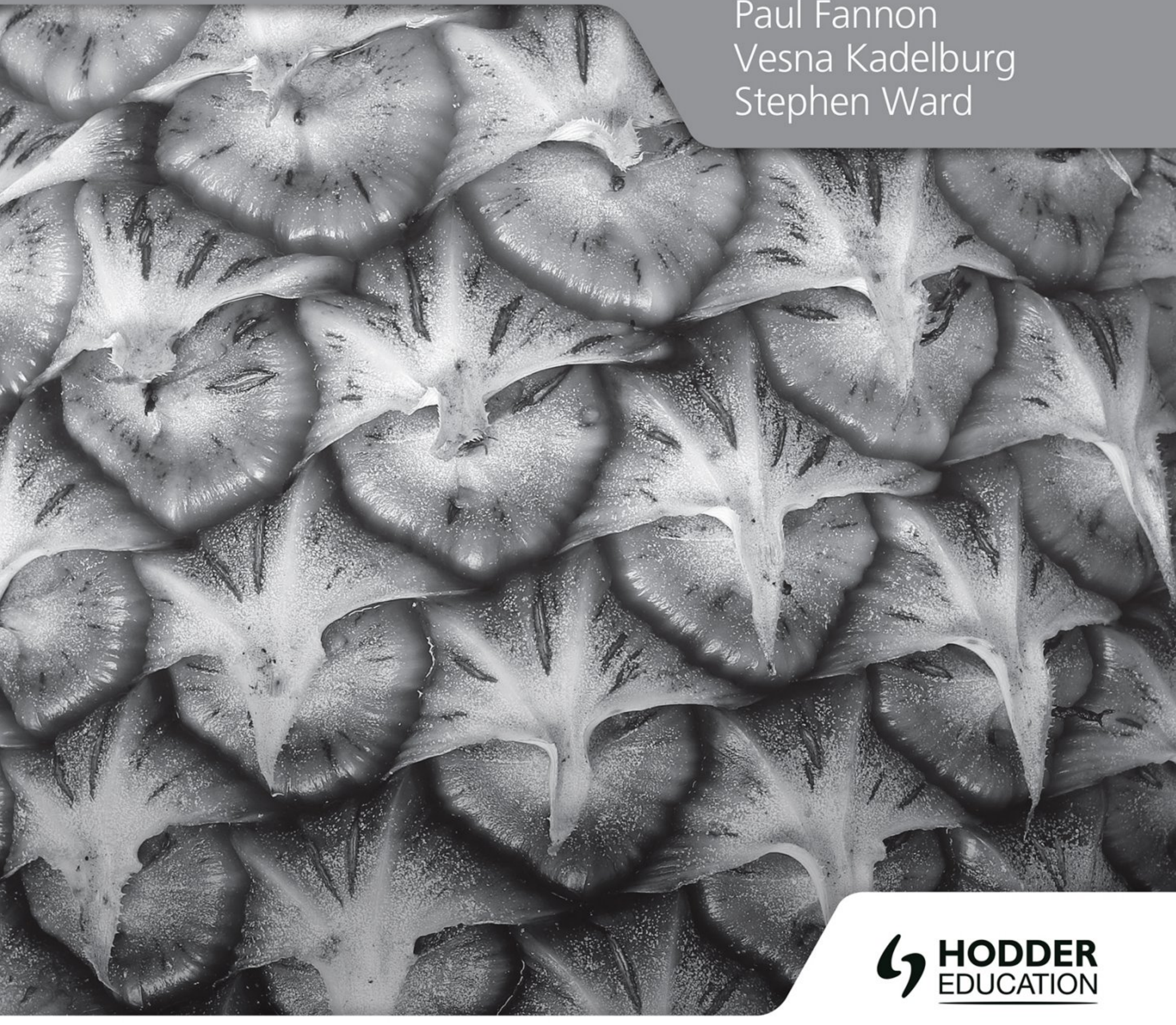


HODDER
EDUCATION

Answers to Practice Questions

1 Number and algebra

1 Just plug the numbers into your calculator. The answer is 4×10^{80}

2 $$\begin{aligned} 3 \times 10^{97} - 4 \times 10^{96} &= 10^{96}(3 \times 10^1 - 4) \\ &= 10^{96}(30-4) \\ &= 26 \times 10^{96} \\ &= 2.6 \times 10 \times 10^{96} \\ &= 2.6 \times 10^{97} \end{aligned}$$

3 We can write this expression as

$$\begin{aligned} \left(\frac{6}{8}\right) \times \left(\frac{10^{30}}{10^{-12}}\right) &= 0.75 \times 10^{30-(-12)} \\ &= 7.5 \times 10^{-1} \times 10^{42} \\ &= 7.5 \times 10^{41} \end{aligned}$$

4 The first term is 20. The common difference is -3. Therefore

$$\begin{aligned} u_{25} &= 20 + (-3) \times (25-1) \\ &= 20 - 3 \times 24 \\ &= -52 \end{aligned}$$

Tip: If this had been a calculator question, you could have just used $u_1 = 20, u_{n+1} = u_n - 3$ in your calculator sequence function.

5 $u_n = 1602 = 21 + 17(n-1)$

$1581 = 17(n-1)$

$93 = n - 1$

$94 = n$

6 $u_4 = 10 = u_1 + 3d$

$u_{10} = 34 = u_1 + 9d$

Subtracting gives;

$24 = 6d$

$4 = d$

Substituting into the first equation:

$10 = u_1 + 3 \times 4$

$u_1 = -2$

Therefore

$$u_{20} = -2 + 19 \times 4$$
$$= 74$$

Tip: If this had been a calculator question, you could have solved the simultaneous equations using your GDC.

7 $u_1 = 13, d = -3$ so

$$S_{30} = \frac{30}{2}\big(2 \times 13 - 3 \times (30 - 1)\big) = -915$$

8 $S_{20} = \frac{20}{2}(4 + 130) = 1340$

9 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 16 + 21 + 26 \ldots$

So the first term is 16 and the common difference is 5.

10 Your GDC should have a sum function, which can be used for this. The answer will be 26350, but you should still write down the first term and common difference found in question 9 as part of your working.

11 This is an arithmetic sequence with first term 500 and common difference 100. The question is asking for S_{28}.

$$S_{28} = \frac{28}{2}(2 \times 500 + 100 \times 27) = 51800 \text{ m}$$

Tip: The hardest part of this question is realising that it is looking for the sum of the sequence, rather than just how far Ahmed ran on the 28th day.

12 2.4% of \$300 is \$7.20. This is the common difference. After one year, there is \$307.20 in the account, so this is the 'first term'.

$u_{10} = 307.20 + 9 \times 7.20 = \$372$

Tip: The hardest part of this question is being careful with what 'after 10 years' means – it is very easy to be out by one year.

13 a The differences in velocity are 1.1, 0.8 and 0.8. Their average is 0.9. When $t = 0.5$ we are looking for the sixth term of the sequence which would be

$u_6 = 0 + 0.9 \times 5 = 4.5 \text{ m s}^{-1}$

b There are many criticisms which could be made about this model – for example:

- There is too little data for it to be reliable.
- There is no theoretical reason given for it being an arithmetic sequence.
- The ball will eventually hit the ground.
- The model predicts that the ball's velocity grows without limit.
- There seems to be a pattern with smaller differences later on.

14 The first term is 32 and the common ratio is $-\frac{1}{2}$.

$$u_{12} = 32 \times \left(-\frac{1}{2}\right)^9 = -\frac{1}{16}$$

15 The first term is 1 and the common ratio is 2 so

$u_n = 1 \times 2^{n-1}$

If $u_n = 4096$ then

$4096 = 2^{n-1}$

There are four ways you should be able to solve this:

- On a non-calculator paper you might be expected to figure out that $4096 = 2^{12}$
- You can take logs of both sides to get $\ln 4096 = (n-1)\ln 2$ and solve for n.
- You can graph $y = 2^{x-1}$ and intersect it with $y = 4096$
- You can create a table showing the sequence and determine which row 4096 is in.

Whichever way, the answer is 13.

16 $u_3 = u_1 r^2 = 16$

$u_7 = u_1 r^6 = 256$

Dividing the two equations:

$$\frac{u_1 r^6}{u_1 r^2} = \frac{256}{16}$$

$r^4 = 16$

$r = \pm 2$

$u_1 = \frac{16}{r^2} = 4$ for both possible values of r.

17 The first term is 162, the common ratio is $\frac{1}{3}$

$$S_8 = \frac{162\left(1 - \frac{1}{3}^8\right)}{1 - \frac{1}{3}} = \frac{6560}{27} \approx 243$$

Tip: You can always use either sum formula, but generally if r is between 0 and 1 the second formula avoids negative numbers.

18 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 10 + 50 + 250 \ldots$

So the first term is 10 and the common ratio is 5.

19 Your GDC should have a sum function, which can be used for this. The answer will be 24414060, but you should still write down the first term and common ratio found in question 18 as part of your working.

20 a This is a geometric sequence with first term 50,000 and common ratio 1.2. 'After 12 days' corresponds to the 13th term of the sequence so:

$u_{13} = 50{,}000 \times 1.2^{12} = 445805$

b The model suggests that the number of bacteria can grow without limit.

21 You could use the formula:

$$FV = 2000 \times \left(1 + \frac{4}{100 \times 12}\right)^{12\times 10} = £2981.67$$

However, the general expectation is that you would use the TVM package on your calculator for this type of question. Make sure you can get the same answer using your package as different calculators have slightly different syntaxes.

22 Using the TVM package, $i = 5.10\%$

23 Unless stated otherwise, you should assume that the compound interest is paid annually. Using $FV = 200$ and $PV = 100$, the TVM package suggests that 33.35 years are required. There 33 years would be insufficient so 34 complete years are required.

24 12% annual depreciation is modelled as compound interest with an interest rate of –12%. Using the TVM package the value is \$24000 to three significant figures.

25 When adjusting for inflation, the 'real' interest rate is $3.2 - 2.4 = 0.8\%$. Using this value in the TVM package we find a final value of \$2081

26 This expression is $2^{-2\times -2} = 2^4 = 16$

27 $(2x)^3 = 2^3 \times x^3 = 8x^3$

28 $10^x = \frac{5}{4}$

This is equivalent to $x = \log_{10}\left(\frac{5}{4}\right)$

29 The given statement is equivalent to

$2x - 6 = \ln 5$

So

$2x = \ln 5 + 6$

$$x = \frac{1}{2}\ln 5 + 3$$

Tip: The answer could be written in several different ways – for example, $\ln(\sqrt{5}\, e^3)$. Generally speaking any correct and reasonably simplified answer would be acceptable.

30 Using appropriate calculator functions:

$\ln 10 + \log_{10} e \approx 2.30 + 0.434 \approx 2.74$

31 LHS $= \frac{m+1}{(m-1)(m+1)} + \frac{m-1}{(m+1)(m-1)}$

$$= \frac{m + 1 + m - 1}{m^2 - m + m - 1}$$

$$= \frac{2m}{m^2 - 1}$$

= RHS

32 a $x - ax = b$

$x(1 - a) = b$

$$x = \frac{b}{1 - a}$$

b Comparing coefficients of x:

$1 = a$

Comparing constant terms:

$0 = b$

Tip: If you are not familiar with comparing coefficients, you can also substitute in $x = 0$ and $x = 1$ to set up some simultaneous equations to get the same results.

33 $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

34 If $x = \log_4 32$ this is equivalent to $4^x = 32$. There are many ways to proceed, but we could write everything in terms of powers of 2:

$(2^2)^x = 2^5$

$2^{2x} = 2^5$

Therefore, $2x = 5$ so $x = 2.5$

35 $x = \log_2(2 \times 5)$
$= \log_2 2 + \log_2 5$
$= 1 + y$

36 $\log_5 12 = \frac{\ln 12}{\ln 5}$

37 $\ln 5^{x-1} = \ln(4 \times 3^{2x})$

$$\begin{aligned}(x-1)\ln 5 &= \ln 4 + \ln 3^{2x}\\ x\ln 5 - \ln 5 &= \ln 4 + 2x\ln 3\\ x\ln 5 - 2x\ln 3 &= \ln 4 + \ln 5\\ x(\ln 5 - 2\ln 3) &= \ln 4 + \ln 5\end{aligned}$$

$$\begin{aligned}x &= \frac{\ln 4 + \ln 5}{\ln 5 - 2\ln 3}\\ &= \frac{\ln 20}{\ln 5 - \ln 9}\\ &= \frac{\ln 20}{\ln\left(\frac{5}{9}\right)}\end{aligned}$$

Tip: In the calculator paper, if an exact form is not required then this type of equation is best solved using an equation solver or a graphical method.

38 The first term is 2, the common ratio is $\frac{1}{3}$, so

$$S_\infty = \frac{2}{1-\frac{1}{3}} = 3$$

39 The common ratio is $2x$. This will converge if $|2x|<1$ which is when $|x| < \frac{1}{2}$. You could also write this as $-\frac{1}{2} < x < \frac{1}{2}$.

40 $$\begin{aligned}(2+x)^4 &= 2^4 + {}^4C_1 \times 2^3 \times x + {}^4C_2 \times 2^2 \times x^2 + {}^4C_3 \times 2^1 \times x^3 + x^4\\ &= 16 + 4 \times 8 \times x + 6 \times 4 \times x^2 + 4 \times 2 \times x^3 + x^4\\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

41 Use $x = 0.01$, then

$$\begin{aligned}(2+0.01)^4 &= 16 + 32 \times 0.01 + 24 \times 0.0001 \ldots \\ &\approx 16 + 0.32 \\ &= 16.32\end{aligned}$$

42 You could find the full expansion, but that would waste time (especially if the brackets were raised to a larger exponent). The general term is

$${}^4C_r(2x)^r\left(-\frac{1}{x^3}\right)^{4-r} = {}^4C_r 2^r(-1)^{4-r}x^r x^{-3(4-r)}$$

$$= {}^4C_r 2^r(-1)^{4-r}x^{4r-12}$$

For this to be a constant, we need $4r - 12 = 0$, so $r = 3$

The term is then ${}^4C_3 2^3(-1)^{4-3} = -32$

43 ${}^8C_3 = \frac{8!}{3!5!}$

$$= \frac{1\times2\times3\times4\times5\times6\times7\times8}{(1\times2\times3)\times(1\times2\times3\times4\times5)}$$

$$= \frac{6\times7\times8}{1\times2\times3}$$

$$= \frac{6\times7\times8}{6}$$

$$= 7\times8 = 56$$

Tip: You could also find this by looking at the appropriate number in Pascal's triangle.

2 Functions

1 Rearrange into the form $y = mx + c$:

$$\begin{aligned}3x - 4y - 5 &= 0 \\ 4y &= 3x - 5 \\ y &= \frac{3}{4}x - \frac{5}{4}\end{aligned}$$

So, $m = \frac{3}{4}, c = -\frac{5}{4}$

2 Use $y - y_1 = m(x - x_1)$:

$$\begin{aligned}y + 4 &= -3(x-2) \\ y + 4 &= -3x + 6 \\ y &= -3x + 2\end{aligned}$$

3 Find the gradient using $m = \frac{y_2 - y_1}{x_2 - x_1}$:

$$m = \frac{1+5}{9+3} = \frac{1}{2}$$

Use $y - y_1 = m(x - x_1)$:

$$y - 1 = \frac{1}{2}(x - 9)$$
$$2y - 2 = x - 9$$
$$x - 2y - 7 = 0$$

4 Gradient of parallel line is $m = 2$

$y - 4 = 2(x - 1)$

$y = 2x + 2$

5 Gradient of perpendicular line is $m = -\frac{1}{-\frac{1}{4}} = 4$

$y - 3 = 4(x + 2)$

$y = 4x + 11$

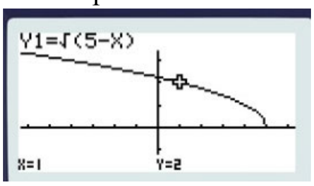
6 Substitute $x = -2$ into the function:

$$f(-2) = 3(-2)^2 - 4$$
$$= 8$$

7 $2x - 1 > 0$

$x > \frac{1}{2}$

8 Graph the function using the GDC:

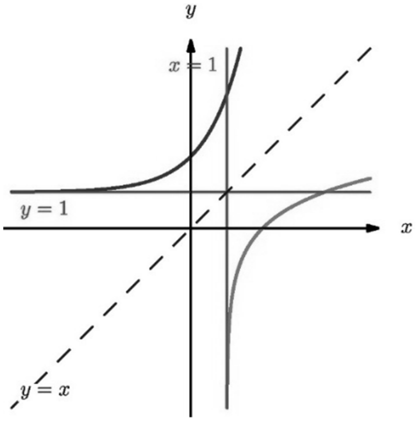


$f(1) = 2$, so range is $f(x) \geq 2$

9 To find $f^{-1}(-8)$, solve $f(x) = -8$:

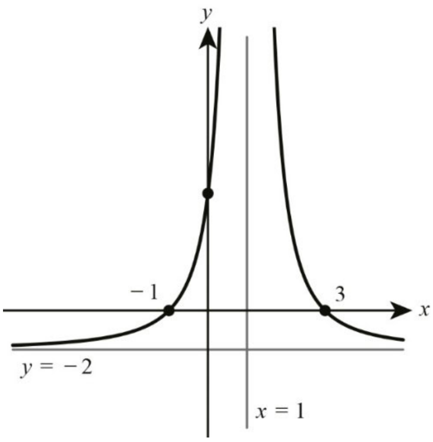
$$4 - 3x = -8$$
$$-3x = -12$$
$$x = 4$$

10 Reflect the graph in the line $y = x$:

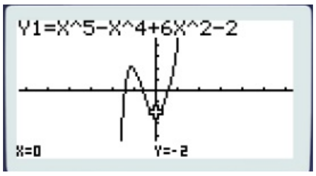


11 Put in the vertical and horizontal asymptotes and the x-intercepts (zeros of the function).

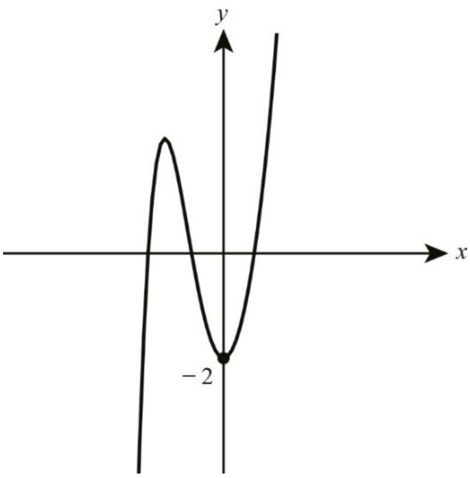
Since $f(x) > -2$, it must tend to ∞ as it approaches the vertical asymptote from either side.



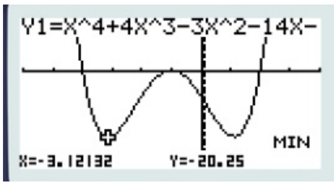
12 Graph the function using the GDC:



The y-intercept is $(0, -2)$. Now sketch the graph from the plot on the GDC:



13 a Graph the function and use ‘min’ and ‘max’ to find the coordinates of the vertices, moving the cursor as necessary:

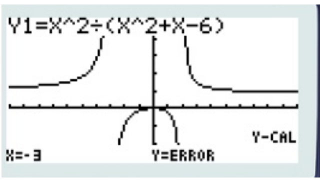


Coordinates of vertices: $(-3.12, -20.3), (-1, 0), (1.12, -20.3)$

b From the graph you can see there is a line of symmetry through the maximum point:

Line of symmetry: $x = -1$

14 Graph the function and look for values where there appear to be asymptotes:



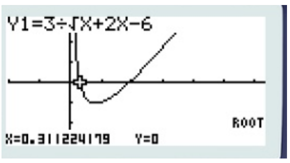
Vertical asymptotes occur at values of x where the y-values appears as ‘error’:

Vertical asymptotes: $x = -3$ and $x = 2$

The y-value approaches 1 as the x value gets big and positive or big and negative:

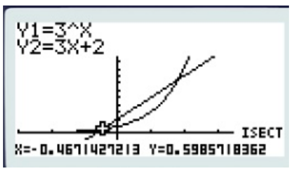
Horizontal asymptote: $y = 1$

15 Graph the function and use ‘root’, moving the cursor from one to the other:



Zeros: $x = 0.311, 1.92$

16 Graph the function and use ‘isct’, moving the cursor from one intersection point to the other:



Points of intersection: $(-0.467, 0.599)$ and $(1.83, 7.50)$

17 a Substitute $\mathrm{g}(x)$ into $\mathrm{f}(x)$:

$$\begin{aligned}\mathrm{f}(\mathrm{g}(x)) &= \frac{1}{(3x-4)-2} \\ &= \frac{1}{3x-6}\end{aligned}$$

b Substitute $\mathrm{f}(x)$ into $\mathrm{g}(x)$:

$$\mathrm{g}(\mathrm{f}(x)) = 3\left(\frac{1}{x-2}\right) - 4$$
$$= \frac{3}{x-2} - 4$$

18 Domain of f is $x \le 2$ so domain of fg is

$$x - 3 \le 2$$
$$x \le 5$$

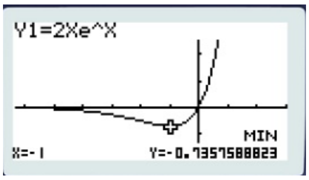
19 Let $y = \mathrm{f}(x)$ and rearrange to make x the subject:

$$y = \frac{x-1}{x+2}$$
$$xy + 2y = x - 1$$
$$x - xy = 1 + 2y$$
$$x(1-y) = 1 + 2y$$
$$x = \frac{1+2y}{1-y}$$

So,

$$\mathrm{f}^{-1}(x) = \frac{1+2x}{1-x}$$

20 Graph the function and use 'min' to find the coordinates of the minimum point:



The turning point has x-coordinate $x = -1$, so largest possible domain of given form is $x \ge -1$.

21 Graph **A** is the only negative quadratic so that is equation **b**.

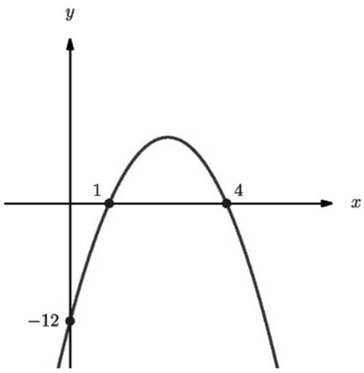
Graph **B** has a negative y-intercept so that is equation **c**.

Graph **C** has a positive y-intercept so that is equation **a**.

22 Negative quadratic with y-intercept $(0, -12)$

For x-intercepts solve $-3x^2 + 15x - 12 = 0$:

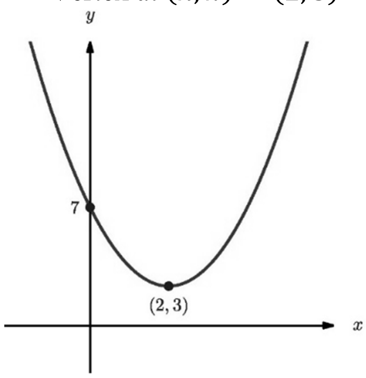
$$-3(x^2 - 5x + 4) = 0$$
$$-3(x-1)(x-4) = 0$$
$$x = 1 \text{ or } 4$$



23 a $x^2 - 4x + 7 = (x-2)^2 - 4 + 7$
$= (x-2)^2 + 3$

b Positive quadratic with y-intercept $(0, 7)$

Vertex at $(h, k) = (2, 3)$



24 $2x^2 + 7x - 15 = 0$
$(2x-3)(x+5) = 0$

So

$2x - 3 = 0$ so $x = \frac{3}{2}$

or

$x + 5 = 0$ so $x = -5$

25 a $x^2 - 5x + 3 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 3$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{13}{4}$$

b Use the completed square form from part **a** and solve for x:

$$x^2 - 5x + 3 = 0$$

$$\left(x - \frac{5}{2}\right)^2 - \frac{13}{4} = 0$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{13}{4}$$

$$x - \frac{5}{2} = \pm\frac{\sqrt{13}}{2}$$

$$x = \frac{5 \pm \sqrt{13}}{2}$$

26 Use the quadratic formula with $a = 3, b = -4, c = -2$:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{40}}{6}$$

$$= \frac{4 \pm 2\sqrt{10}}{6}$$

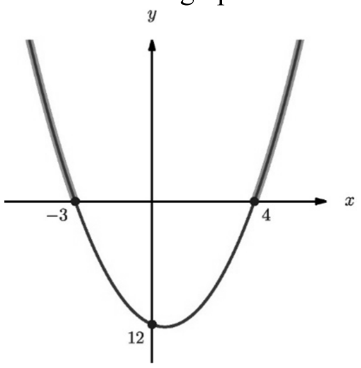
$$= \frac{2 \pm \sqrt{10}}{3}$$

27 Solve the equation $x^2 - x - 12 = 0$:

$(x-4)(x+3) = 0$

$x = 4 \text{ or } -3$

Sketch the graph:



So, $x < -3$ or $x > 4$

28 $\Delta = 5^2 - 4(4)(3)$

$= 25 - 48$

$= -23 < 0$

So, no real roots

29 Two distinct real roots, so $\Delta > 0$ (where $a = 3k, b = 4, c = 12k$)

$4^2 - 4(3k)(12k) > 0$

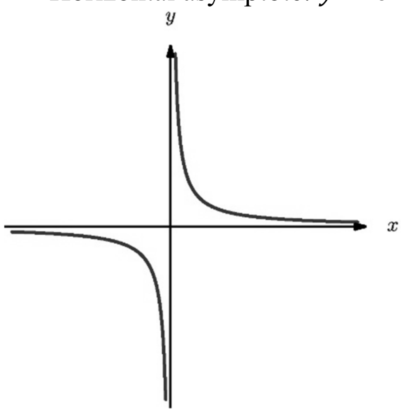
$4 - 36k^2 > 0$

$k^2 < \frac{1}{9}$

$-\frac{1}{3} < k < \frac{1}{3}$

30 Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

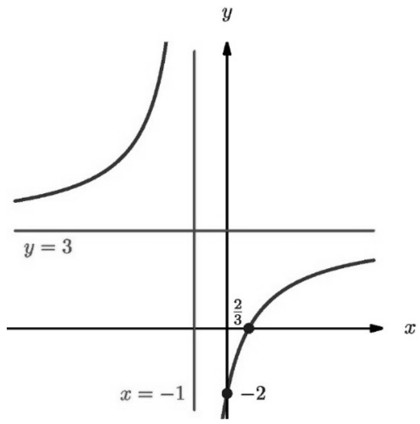


31 $y = \frac{ax+b}{cx+d}$ has

x-intercept $\left(-\frac{b}{a}, 0\right)$ y-intercept $\left(0, \frac{b}{d}\right)$

Vertical asymptote $x = -\frac{d}{c}$

Horizontal asymptote $y = \frac{a}{c}$



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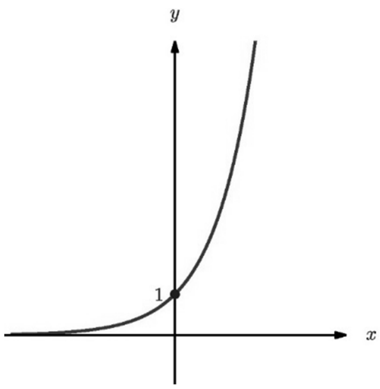

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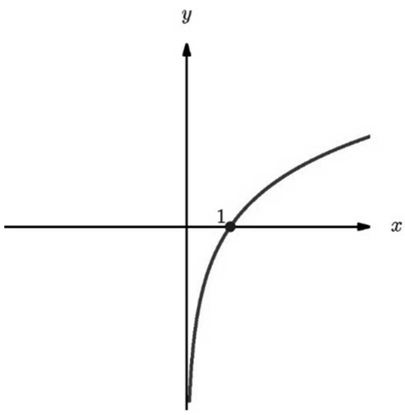
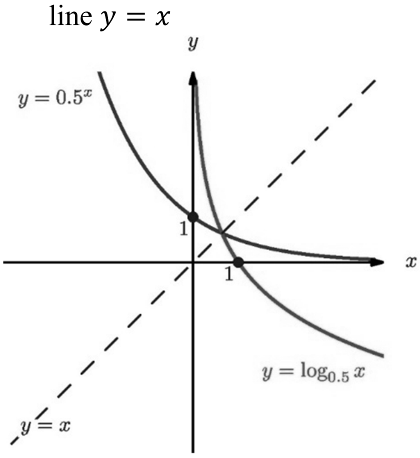


34 $y = 0.5^x$ is decreasing since $0.5 < 1$.

$y = 0.5^x$ and $y = \log_{0.5} x$ are inverse functions, so one is a reflection of the other in the line $y = x$



35 $2.8^x = e^{x \ln 2.8}$

$= e^{1.03x}$

So, $k = 1.03$

36 Rearrange so RHS is 0 and then factorise:

$x \ln x - 4x = 0$

$x(\ln x - 4) = 0$

So

$x = 0$

or

$\ln x = 4$

$x = e^4$

37 Let $y = \sqrt{x}$

$y^2 - 7y + 10 = 0$

$(y - 2)(y - 5) = 0$

$y = 2 \text{ or } 5$

So,

$\sqrt{x} = 2 \text{ or } 5$

$x = 4 \text{ or } 25$

38 Combine the log terms using $\log x + \log y = \log xy$ and then undo the log leaving a quadratic equation:

$\log_2(x(x+2)) = 3$

$x^2 + 2x = 2^3$

$x^2 + 2x - 8 = 0$

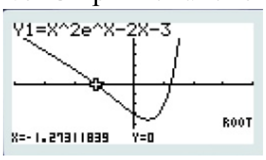
$(x + 4)(x - 2) = 0$

$x = -4 \text{ or } 2$

However, checking both possible solutions in the original equation, you can see that $x = -4$ is not valid, as you cannot have a log of a negative number.

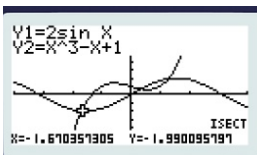
So, $x = 2$.

39 Graph the function and use 'root', moving the cursor from one to the other:



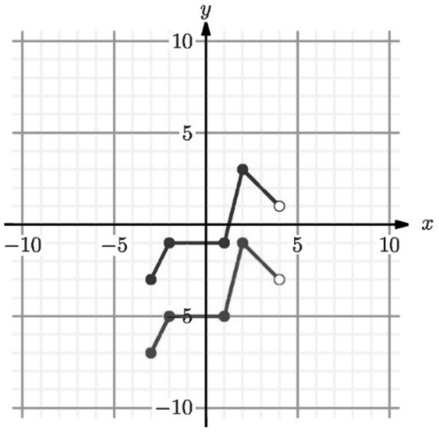
$x = -1.27 \text{ or } 1.25$

40 This could be solved as above by rearranging to the form $f(x) = 0$ or by finding the intersection of the curves $y = 2\sin x$ and $y = x^3 - x + 1$:



$x = -1.67, 0.353 \text{ or } 1.31$

41 $y = \mathrm{f}(x) - 4$ is a vertical translation by -4:



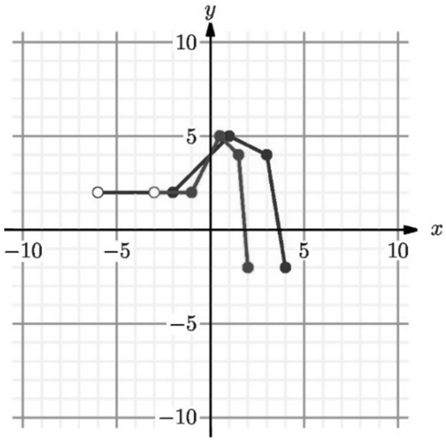
42 A translation 3 units to the right is given by $y = \mathrm{f}(x - 3)$:

$$\begin{aligned} y &= (x-3)^2 - 2(x-3) + 5 \\ &= x^2 - 6x + 9 - 2x + 6 + 5 \\ &= x^2 - 8x + 20 \end{aligned}$$

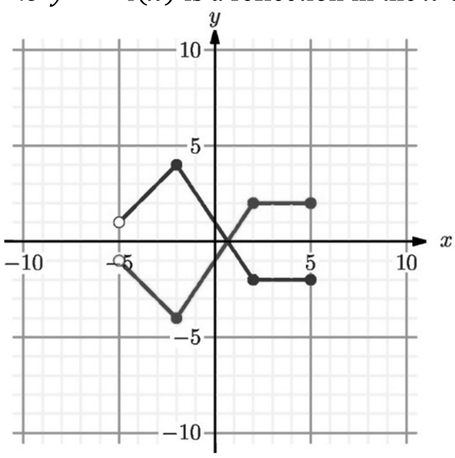
43 A vertical stretch by scale factor 2 is given by $y = 2\mathrm{f}(x)$:

$$\begin{aligned} y &= 2(3x^2 + x - 2) \\ &= 6x^2 + 2x - 4 \end{aligned}$$

44 $y = \mathrm{f}(2x)$ is a horizontal stretch with scale factor $\frac{1}{2}$:



45 $y = -\mathrm{f}(x)$ is a reflection in the x-axis:



46 A reflection in the y-axis is given by $y = \mathrm{f}(-x)$:

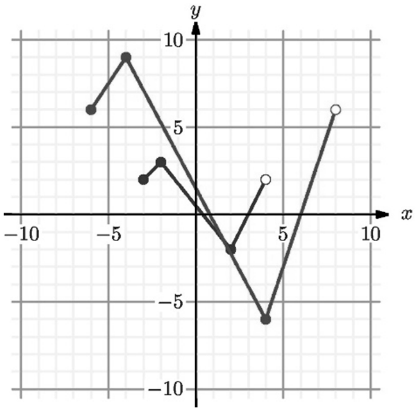
$$y = (-x)^3 + 3(-x)^2 - 4(-x) + 1$$
$$= -x^3 + 3x^2 + 4x + 1$$

47 $y = 4\mathrm{f}(x) + 1$ is a vertical stretch with scale factor 4 followed by a vertical translation by 1.

So, $(3, -2) \longrightarrow (3, -2 \times 4 + 1) = (3, -7)$

Note that the order matters for two vertical transformations.

48 $y = 3\mathrm{f}\left(\frac{1}{2}x\right)$ is vertical stretch with scale factor 3 and a horizontal stretch with scale factor 2 (in either order):



3 Geometry and trigonometry

1 $d = \sqrt{(7-2)^2 + (3-(-4))^2 + (-1-5)^2}$
$= \sqrt{25 + 49 + 36}$
$= 10.5$

2 $M = \left(\frac{1+(-5)}{2}, \frac{8+2}{2}, \frac{-3+4}{2}\right)$
$= (-2, 5, 0.5)$

3 Radius = 8 cm

Volume $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 8^3$

$= 2140 \text{ cm}^3$

Surface area $= 4\pi r^2$

$= 4\pi \times 8^2$

$= 804 \text{ cm}^2$

4 Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 6^2 \times 15$

$= 565 \text{ cm}^3$

Slope length, l, is given by:

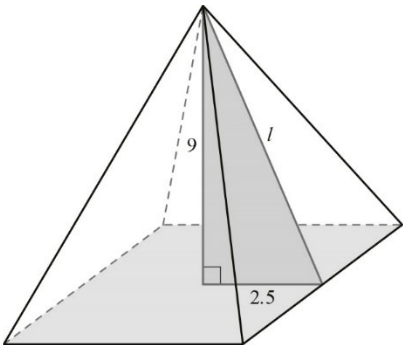
$$l = \sqrt{r^2 + h^2}$$
$$= \sqrt{6^2 + 15^2}$$
$$= 3\sqrt{29}$$

Surface area $= \pi r l + \pi r^2$

$$= \pi \times 6 \times 3\sqrt{29} + \pi \times 6^2$$
$$= 418 \text{ cm}^2$$

5 Volume $= \frac{1}{3}x^2 h$

$$= \frac{1}{3} \times 5^2 \times 9$$
$$= 75.0 \text{ cm}^3$$



$$l = \sqrt{2.5^2 + 9^2}$$
$$= \frac{\sqrt{349}}{2}$$

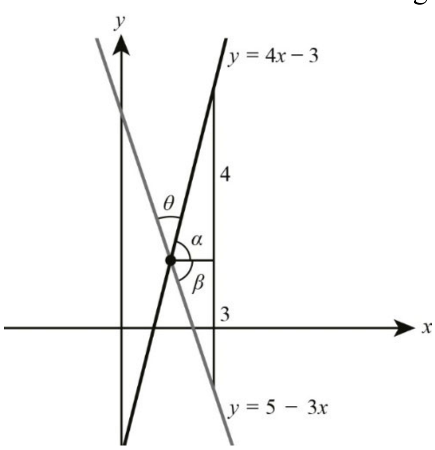
Surface area $= 5^2 + 4\left(\frac{1}{2} \times 5 \times \frac{\sqrt{349}}{2}\right)$

$$= 118 \text{ cm}^2$$

6 Volume $= \pi r^2 h + \frac{2}{3}\pi r^3$

$$= \pi \times 5^2 \times 30 + \frac{2}{3}\pi \times 5^3$$
$$= 2620 \text{ m}^3$$

7 Draw the lines and label the angle required θ:



Since the gradient of y = 4x − 3 is 4,

$\tan\alpha = \frac{4}{1}$

$\alpha = \tan^{-1} 4 = 76.0°$

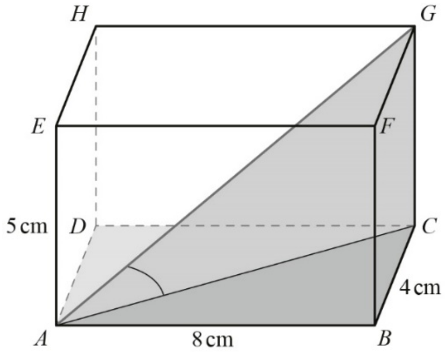
Since the gradient of $y = 5 - 3x$ is -3,

$\tan\beta = \frac{3}{1}$

$\alpha = \tan^{-1} 3 = 71.6°$

So, $\theta = 180 - 76.0 - 71.6 = 32.4°$

8 Draw in the diagonal AG. Angle needed is $G\hat{A}C = \theta$



First work in triangle ABC:

$AC = \sqrt{8^2 + 4^2}$

$= \sqrt{80}$

Then in triangle ACG:

$\tan\theta = \frac{5}{\sqrt{80}}$

$\theta = \tan^{-1}\frac{5}{\sqrt{80}} = 29.2°$

9 Draw in the two diagonals – they will intersect at the midpoint of each, M.

From triangle AGC:

$AG = \sqrt{\left(\sqrt{80}\right)^2 + 5^2}$

$= \sqrt{105}$

By symmetry, $EC = \sqrt{105}$

And $AM = CM = \frac{\sqrt{105}}{2}$

Using the cosine rule in triangle AMC:

$\cos M = \frac{AM^2 + CM^2 - AC^2}{2(AM)(CM)}$

$= \frac{26.25 + 26.25 - 80}{52.5}$

$M = 121.6°$

So, acute angle between AG and EC is $180 - 121.6 = 58.4°$

10 $\sin\theta = \frac{1.8}{4.9}$

$$\theta = \sin^{-1}\frac{1.8}{4.9} = 21.6°$$

11 By the sine rule,

$$\frac{3.8}{\sin 80} = \frac{AC}{\sin 55}$$
$$AC = \frac{3.8}{\sin 80} \times \sin 55 = 3.16 \text{ cm}$$

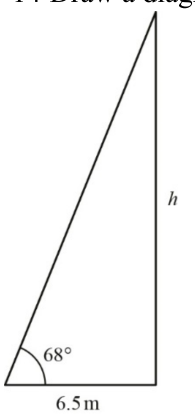
12 By the cosine rule,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}$$
$$C = \cos^{-1}\left(\frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}\right) = 47.2°$$

13 $A = \frac{1}{2} \times 6 \times 15 \times \sin 42$

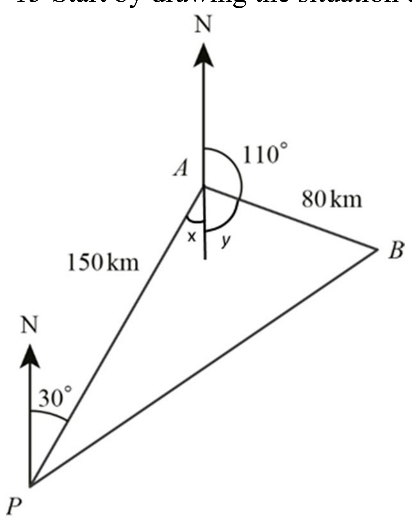
$= 30.1$

14 Draw a diagram:



$$\tan 68 = \frac{h}{6.5}$$
$$h = 6.5 \tan 68 = 16.1 \text{ m}$$

15 Start by drawing the situation described:



$x = 30°$ by alternate angles

$y = 180 - 110 = 70°$

So, $P\hat{A}B = 30 + 70 = 100°$

By the cosine rule,

$$d^2 = 150^2 + 80^2 - 2 \times 150 \times 80 \times \cos 100$$
$$d = \sqrt{150^2 + 80^2 - 2 \times 150 \times 80 \times \cos 100}$$
$$= 182 \text{ km}$$

16 a $55° = 55 \times \frac{2\pi}{360} = \frac{11\pi}{36}$ radians

b 1.2 radians $= 1.2 \times \frac{360}{2\pi} = 68.8°$

17 $s = r\theta$

$= 6 \times 0.7$

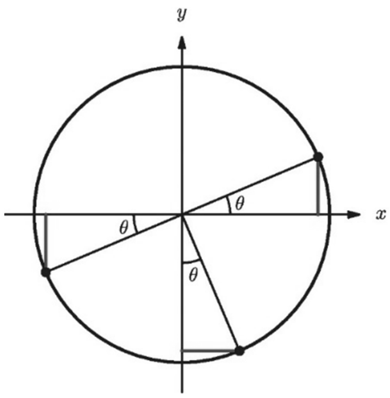
$= 4.2$ cm

18 $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 10^2 \times 1.8$

$= 90 \text{ cm}^2$

19



Using the unit circle:

a $\sin(\theta + \pi) = -\sin\theta = -0.4$

b $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta = 0.4$

20 $\tan(2\pi - \theta) = \frac{\sin(2\pi-\theta)}{\cos(2\pi-\theta)}$

$= \frac{\sin(-\theta)}{\cos(-\theta)}$ (since sin and cos are 2π periodic)

$= \frac{-\sin\theta}{\cos\theta}$

$= -\tan\theta$

Note that certain relationships, such as $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, are used so often that you should just know them. You don't want to have to go back to the unit circle each time to derive them.

21 Relate $\frac{4\pi}{3}$ to one of the 'standard' angles: $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}\cos\frac{4\pi}{3} &= \cos\left(\frac{\pi}{3} + \pi\right)\\ &= -\cos\frac{\pi}{3} \qquad \text{(using the unit circle or remembering } \cos(\theta + \pi) = -\cos\theta\text{)}\\ &= -\frac{1}{2}\end{aligned}$$

22 By the sine rule,

$$\frac{\sin\theta}{14} = \frac{\sin 38}{11}$$

$$\theta = \sin^{-1}\left(\frac{\sin 38}{11} \times 14\right)$$

$\theta = 51.6°$ or $\theta = 180 - 51.6 = 128.4°$

Check that each value of θ is possible by making sure that the angle sum in each case is less than 180°:

$51.6 + 38 = 89.6 < 180$

$128.4 + 38 = 166.4 < 180$

So both are possible:

$\theta = 51.6°$ or $128.4°$

23 Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to relate the value of $\cos\theta$ to the value of $\sin\theta$:

$$\begin{aligned}\cos^2\theta &= 1 - \sin^2\theta\\ &= 1 - \frac{9}{16}\\ &= \frac{7}{16}\end{aligned}$$

$$\cos\theta = \pm\frac{\sqrt{7}}{4}$$

But $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \pi$

So, $\cos\theta = -\frac{\sqrt{7}}{4}$

24 $$\begin{aligned}(\cos\theta + \sin\theta)^2 &\equiv \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta\\ &\equiv 2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta\\ &= \sin 2\theta + 1\end{aligned}$$

25 Relate 15° to a 'standard' angle:

$\cos(2 \times 15) = \cos 30 = \frac{\sqrt{3}}{2}$

So, use $\cos 2\theta = 2\cos^2\theta - 1$:

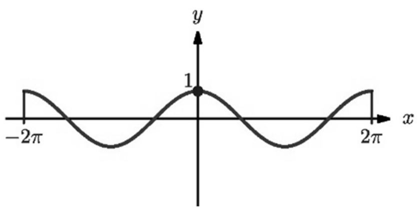
$\cos 30 = 2\cos^2 15 - 1$

$$\cos^2 15 = \frac{\cos 30 + 1}{2} = \frac{\frac{\sqrt{3}}{2} + 1}{2} = \frac{\sqrt{3} + 2}{4}$$

So, taking the positive square root (since $\cos 15° > 0$):

$$\cos 15° = \sqrt{\frac{\sqrt{3} + 2}{4}}$$

26

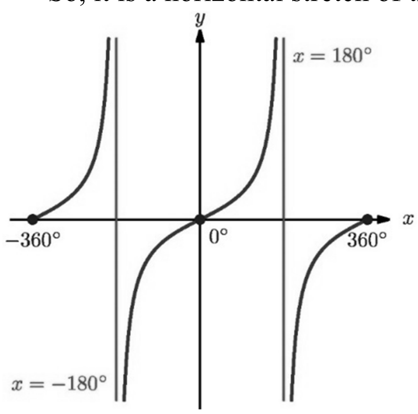


Period = 2π

Amplitude = 1

27 $y = \tan\frac{x}{2}$ is a transformation of the form $y = \mathrm{f}\left(\frac{1}{2}x\right)$ where $\mathrm{f}(x) = \tan x$

So, it is a horizontal stretch of the tan curve with scale factor 2.



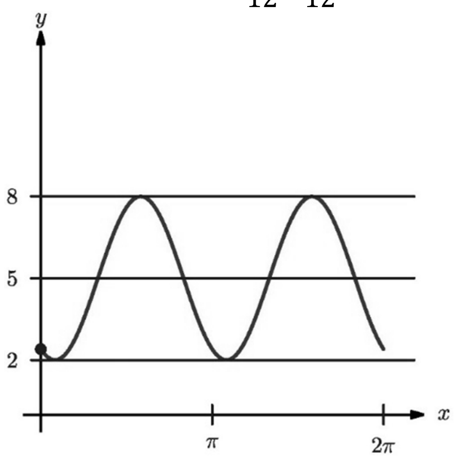
28 $y = 3\sin\theta + 5$ is a transformation of the form $y = 3f(\theta) + 5$ where $f(\theta) = \sin\theta$

So, vertically there is a stretch with scale factor 3 followed by a translation by 5 of the sin curve.

Horizontally, max points of $\sin\theta$ occur at $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \dots$

So,

$$2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}.$$
$$x - \frac{\pi}{3} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \dots$$
$$x = \frac{7\pi}{12}, \frac{19\pi}{12}$$



29 The amplitude is half the difference between the minimum and maximum heights:

$$a = \frac{0.5 - 0.16}{2} = 0.17$$

The period is twice the time from the maximum height to the minimum height:

$\text{Period} = 2 \times 0.3 = 0.6$

But period $= \frac{2\pi}{b}$

So,

$$0.6 = \frac{2\pi}{b}$$
$$b = \frac{10\pi}{3}$$

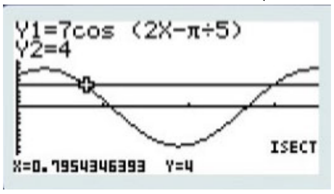
c is halfway between the maximum and minimum heights:

$$c = \frac{0.5 + 0.16}{2} = 0.33$$

So,

$$h = 0.17\cos\left(\frac{10\pi}{3}t\right) + 0.33$$

30 Graph $y = 7\cos\left(2x - \frac{\pi}{5}\right)$ and $y = 4$ and find the x-values of the intersection points:

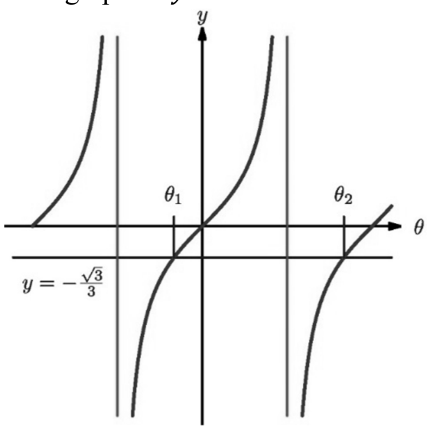


$x = 0.795, 2.97$

31 Since $\tan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ and $\tan(-\theta) = -\tan\theta$,

$$\theta_1 = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

The graph of $y = \tan\theta$ for $-\pi < \theta < \pi$ shows there is one other solution:



$$\theta_2 = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$$

So, $\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$

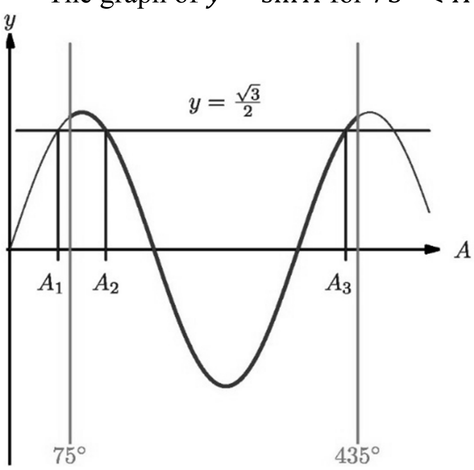
32 Let $A = x + 75$

Then, $0 < x < 360 \Rightarrow 75 < x + 75 < 360 + 75$.

So, $\sin A = \frac{\sqrt{3}}{2}$ for $75° < A < 435°$

$$A_1 = \sin^{-1}\frac{\sqrt{3}}{2} = 60°$$

The graph of $y = \sin A$ for $75° < A < 435°$ shows there are two solutions (A_2 and A_3):



$A_2 = 180 - 60 = 120°$
$A_3 = 60 + 360 = 420°$

So, $A = 120°, 420°$

$x = 45°, 345°$

33 Use the sine double angle formula so that everything is a function of θ:

$$\sin 2\theta = \sin\theta$$
$$2\sin\theta\cos\theta = \sin\theta$$
$$2\sin\theta\cos\theta - \sin\theta = 0$$
$$\sin\theta\,(2\cos\theta - 1) = 0$$

So,

$\sin\theta = 0$
$\theta = 0, \pi, 2\pi$

or

$2\cos\theta - 1 = 0$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$

Note that while it is tempting to cancel $\sin\theta$ on the second line of the working, this would lose the solutions resulting from $\sin\theta = 0$.

34 Use $\sin^2 x \equiv 1 - \cos^2 x$ so that only $\cos x$ is involved:

$$2\sin^2 x - 3\cos x - 3 = 0$$
$$2(1 - \cos^2 x) - 3\cos x - 3 = 0$$
$$2 - 2\cos^2 x - 3\cos x - 3 = 0$$
$$2\cos^2 x + 3\cos x + 1 = 0$$

This is a quadratic in $\cos x$:

$(2\cos x + 1)(\cos x + 1) = 0$

So

$$\cos x = -\frac{1}{2}$$
$$x = 120°, -120°$$

or

$\cos x = -1$
$x = 180°, -180°$

$x = -180°, -120°, 120°, 180°$

4 Statistics and probability

1 a Discrete – it can only take certain values.

b Continuous (although its measurement might be discrete since it will be to a particular accuracy)

c Discrete

2 a There are many possibilities. It could be all of his patients, all of his ill patients, all people in his area, or all people in the world.

b This is not a random sample of the population since there are some people (those who do not attend the clinic) who cannot possibly be included.

3 Yes – people who do less exercise might be less likely to choose to participate.

4 Yes – there appears to be consistency when the observation is repeated (within a reasonable statistical noise).

5 Item D is not a possible human height. It might have been a participant not taking the test seriously or it might have been a misread (e.g. giving height in feet and inches). You could either return to participant D and ask them to check their response, or if this were not possible you would discard the data item.

6 The IQR is 4. Anything above $11 + 1.5 \times 4 = 17$ is an outlier. Just because a data item is an outlier does not mean that it should be excluded. It should be investigated carefully to ensure that it is still a valid member of the population of interest.

7 Convenience sampling.

8 The proportion from Italy is $\frac{60}{150} = \frac{2}{5}$. The stratified sample must be in the same proportion, so it should contain $\frac{2}{5} \times 20 = 8$ students from Italy.

9 The proportion is $\frac{18+12}{15+18+12} = \frac{30}{45} = \frac{2}{3}$

10 This is all of the 30 to 40 group and half of the 20 to 30 group, which is $7 + \frac{4}{2} = 9$

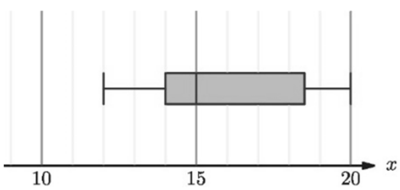
11 a The total frequency is 160. Reading off half of this frequency (80) on the frequency axis is about 42 on the x-axis, which is the median.

b The lower quartile corresponds to a frequency of 40, which is an x-value of approximately 30.

The upper quartile corresponds to a frequency of 120 which is an x-value of approximately 60. Therefore, the interquartile range is $60 - 30 = 30$

c The 90th percentile corresponds to a frequency of $0.9 \times 160 = 144$. This has an x-value of about 72 which is the 90th percentile.

12 Putting the data into the GDC, the following summary statistics can be found:



Min: 12, lower quartile: 14, median: 16, upper quartile: 18.5, max: 20

13 a Both have the similar spread (same IQR (4) and range excluding outliers (10) but population A is higher on average (median of 16 versus 10).

b B is more likely to be normally distributed as it has a symmetric distribution.

14 The mode is 14 as it is the only one which occurs twice. From the GD, the median is 18 and the mean is 18.5

Tip: You should also be able calculate the median without a calculator.

15 The mean is $\frac{23+x}{5} = 7$

So $23 + x = 35$, therefore $x = 12$

16 a The midpoints are 15, 25, 40, 55.

$n = 10 + 12 + 15 + 13 = 50$

$\bar{x} = \frac{10\times15+12\times25+15\times40+13\times55}{50} = \frac{1765}{50} = 35.3$

b We are not using the original data.

17 The modal class is $15 < x \leq 20$

18 a From the GDC, $Q_3 = 18, Q_1 = 7$ so $\text{IQR} = 11$

b From the GDC, the standard deviation is 5.45

c The variance is $5.45^2 = 29.7$

19 The new mean is $12 \times 2 + 4 = 28$

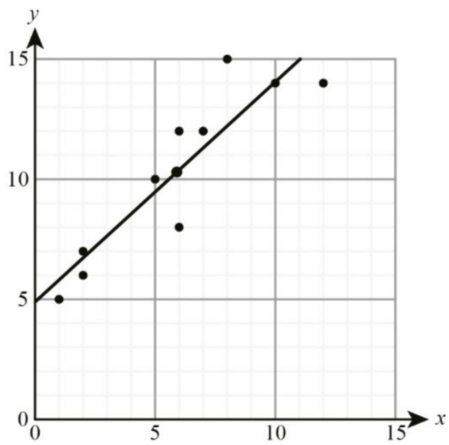
The new standard deviation is $10 \times 2 = 20$.

20 Using the GDC, the lower quartile is 16 and the upper quartile is 28.5

21 a From the GDC: $r = 0.910$

b There is strong positive correlation between x and y

22 a Something like:



b Approximately 7.5

23 From GDC, $y = 0.916x + 4.89$

24 a (i) 13.1

(ii) 23.21

(iii) 5.58

b Only (**i**). Part (**ii**) is extrapolation and part (**iii**) is using a y-on-x line inappropriately.

25 a This is the expected number of text messages sent by a pupil who does not spend any time on social media in a day.

b For every additional hour spent on social media, the model predicts that the pupil will send 1.4 additional texts.

26 a Split the data into the first four points and the next five points and do a regression for each part separately.

$$L = \begin{cases} 4.19A - 0.259 & A < 6 \\ 0.830A + 25.4 & A > 6 \end{cases}$$

b Using the first part of the piecewise graph:

$L = 4.19 \times 3 - 0.259 = 12.3$

So expect a length of 12.3cm

27 $\frac{134}{200} = 0.67$

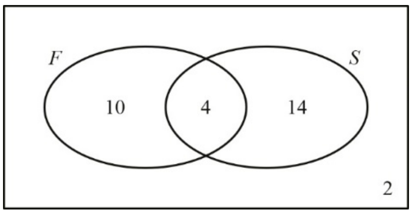
28 There are six possible outcomes of which three (2, 3 and 5) are prime, so the probability is $\frac{3}{6} = 0.5$

29 $\mathrm{P}(A') = 1 - \mathrm{P}(A) = 0.4$

30 $30 \times 0.05 = 1.5$

Tip: Remember that expected values should not be rounded to make them actually achievable.

31 We can illustrate this in a Venn diagram:



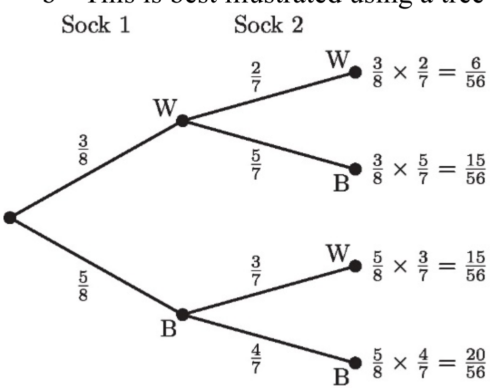
There are $14 - 4 = 10$ students who study only French

There are $18 - 4 = 14$ students who study only Spanish

Therefore, there are $10 + 14 + 4 = 28$ students who study either French or Spanish. This leaves 2 students who do not study either, so the probability is $\frac{2}{30} = \frac{1}{15}$

32 a There will be 7 socks left, of which 5 are black so it is $\frac{5}{7}$

b This is best illustrated using a tree diagram:



There are two branches relevant to the question.

White then black:

$$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

Black then white:

$$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$$

The total probability is $\frac{30}{56} = \frac{15}{28}$

33 a This can be best illustrated using a sample space diagram:

		1st Roll			
		1	**2**	**3**	**4**
2nd Roll	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

There are 16 places in the sample space diagram: 6 of them have a score above 5 (shaded in the diagram) therefore the probability is $\frac{6}{16} = \frac{3}{8}$

b In the sample space diagram, there are 6 scores above 5. Two of them are 7 so $\mathrm{P}(\text{score} = 7|\text{score} > 5) = \frac{2}{6} = \frac{1}{3}$

34 a $x = 100 - 40 - 30 - 20 = 10$

b There are 60 out of 100 students who prefer Maths, so $\frac{60}{100} = \frac{3}{5}$

35 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.5 + 0.7 - 0.3 = 0.9$

36 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - 0 = 0.6$

37 There are 70 people who prefer soccer. Out of these 40 prefer maths. So $\mathrm{P}(\text{Maths}|\text{Soccer}) = \frac{40}{70} = \frac{4}{7}$

38 $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.24$

39 W can take three possible values: 0, 1 or 2

$$\mathrm{P}(W=0)=\mathrm{P}(BB)=\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}$$

$$\mathrm{P}(W=1)=\mathrm{P}(BW)+\mathrm{P}(WB)=\frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6}=\frac{4}{7}$$

$$\mathrm{P}(W=2)=\mathrm{P}(WW)=\frac{3}{7}\times\frac{2}{6}=\frac{1}{7}$$

So

$\boldsymbol{w}$	0	1	2
$\mathbf{P}(\boldsymbol{W}=\boldsymbol{w})$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

40 Create a table:

$\boldsymbol{x}$	0	1	2
$\mathbf{P}(\boldsymbol{X}=\boldsymbol{x})$	k	$2k$	$3k$

The total probability is $k+2k+3k=6k$ which must equal 1 so $k=\frac{1}{6}$

41 $E(X)=0.5\times 0.5+1\times 0.4+2.5\times 0.1=0.9$

42 a The probability of winning a prize is $0.095+0.005=0.1$. Out of this, the probability of winning \$2000 is 0.01, so the conditional probability is $\frac{0.005}{0.1}=0.05$

b $E(X)=0\times 0.9+10\times 0.095+2000\times 0.005=10.95$

$\mathrm{P}(X>10.95)=\mathrm{P}(X=2000)=0.005$

43 $E(X)=-1\times 0.6+0\times 0.3+0.1k=0.1k-0.6$

If the game is fair then $E(X)=0$ so:

$0.1k-0.6=0$

$0.1k=0.6$

$k=6$

44 The outcome of each trial is not independent of the previous trial.

45 Your calculator should have two functions – one which finds $\mathrm{P}(X=x)$, which we will use in part **a** and one which finds $\mathrm{P}(X\leq x)$ which we will use in part **b**.

a $\mathrm{P}(X=2)=0.3456$

b To use the calculator, you need to write the given question into a cumulative probability:

$$\mathrm{P}(X\geq 3)=1-\mathrm{P}(X\leq 2)=1-0.683=0.317$$

46 a If X is the number of heads then $E(X)=np=10\times 0.6=6$

b $\mathrm{Var}(X)=np(1-p)=10\times 0.6\times 0.4=2.4$ so the standard deviation is $\sqrt{2.4}\approx 1.55$

47 We know that about 68% of the data occurs within one standard deviation of the mean, but this takes us to negative values of time which is not possible.

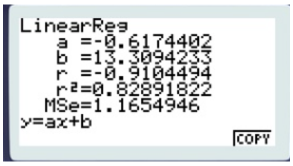
48 a It is a symmetric, bell-shaped curve.

b The line of symmetry is approximately at 50, so this is a good estimate of the mean.

49 $\mathrm{P}(11 < X < 15)$ can be found on the calculator – either directly or as $\mathrm{P}(X < 15) - \mathrm{P}(X < 11)$. It equals 0.625

50 Some calculators can deal with the given information directly, but some require you to first convert the information into a cumulative probability: $\mathrm{P}(X \le k) = 0.3$. Using the inverse normal function on the calculator gives $k = 92.1$

51 Enter the x-data in the y column and the y-data in the x column. Remember that the output is in the form $x = ay + b$:



Regression line is:

$x = -0.617y + 13.3$

52 a Substitute each value of y into the equation:

(i) $x = 1.82 \times 20 - 11.5 = 24.9$

(ii) $x = 1.82 \times 35 - 11.5 = 52.2$

b The prediction when $y = 20$ can be considered as reliable since 20 is within the range of known y-values and the correlation coefficient is close to 1 suggesting a good linear relationship.

The prediction when $y = 35$ cannot be considered as reliable since the relationship needs to be extrapolated significantly beyond the range of the given data.

53 Use the standard formula with A and B swapped around. Note that $A \cap B$ is the same as $B \cap A$:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{0.4}{0.6} = \frac{2}{3}$$

54 A and B are independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ so first find $\mathrm{P}(A \cap B)$:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A) - \mathrm{P}(A \cap B)$$
$$0.7 = 0.2 + 0.8 - \mathrm{P}(A \cap B)$$
$$\mathrm{P}(A \cap B) = 0.3$$

$\mathrm{P}(A)\mathrm{P}(B) = 0.2 \times 0.8 = 0.16$

$\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\mathrm{P}(B)$

So A and B are not independent

55 The z-value is the number of standard deviations from the mean:

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{17-10}{4.8}$$
$$= 1.46$$

17 is 1.46 standard deviations from the mean.

56 Since the mean and variance are unknown, write each probability statement in terms of the standard normal $Z \sim N(0, 1)$ and use the inverse normal to find z_1 and z_2:

$$\mathrm{P}(X < 12) = 0.3$$
$$\mathrm{P}(Z < z_1) = 0.3$$
$$z_1 = -0.52440$$

$$\mathrm{P}(X > 34) = 0.2$$
$$\mathrm{P}(Z > z_2) = 0.2$$
$$z_2 = 0.84162$$

Use $z = \frac{x-\mu}{\sigma}$ to form two equations in μ and σ:

$$-0.52440 = \frac{12-\mu}{\sigma}$$
$$\mu - 0.52440\sigma = 12 \quad (1)$$

$$0.84162 = \frac{34-\mu}{\sigma}$$
$$\mu + 0.84162\sigma = 34 \quad (2)$$

Solve (1) and (2) simultaneously on the GDC:

$\mu = 20.4, \sigma = 16.1$

5 Calculus

1

x	$\frac{\sin(3x)}{0.2x}$
10	0.25
5	0.2588
1	0.2617
0.1	0.2618

The limit is 0.26

2 Look at the values on the graph close to $x = 2$.

The limit is 0.5

3 The derivative is $\frac{dy}{dx} = 12 - 5 = 7$.

4 'rate' means $\frac{dA}{dt}$; 'decreases' means that the rate of change is negative.

$$\frac{dA}{dt} = -kA$$

5 The y value is $f(x)$ and the gradient is $f'(x)$. So when $y = 4$, $f'(x) = -1$.

6 Use $\Delta x = x_Q - 4$, $\Delta y = y_Q - 2$ and gradient $= \frac{\Delta y}{\Delta x}$

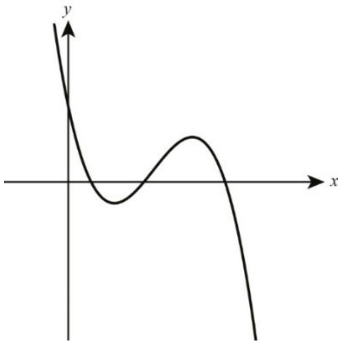
x_Q	y_Q	Δx	Δy	Gradient of PQ
5	2.236	1	0.236	0.236
4.1	2.025	0.1	0.025	0.248
4.01	2.002	0.01	0.002	0.250
4.001	2.000	0.001	0.000	0.250

The gradient is ≈ 0.25

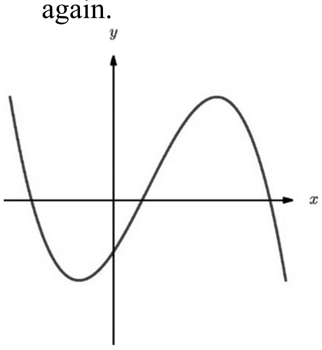
7 $f'(x)$ is where the graph is decreasing, which is between the two turning points.

$-1.29 < x < 1.29$

8 The gradient starts positive but decreasing, then changes to negative, then back to positive and then to negative again.



9 The gradient starts off negative, so $f(x)$ is decreasing. It then increases, and then decreases again.



10 $\frac{dy}{dx} = 8x + \frac{1}{2}x^{-6} - 3$

11 a $f(x) = 12x^2 - 3x^6$ so $f'(x) = 24x - 18x^5$

b $f(x) = 1 - \frac{3}{2}x^{-4}$, so $f'(x) = \frac{6}{x^5}$ [or $6x^{-5}$]

c $f(x) = \frac{4}{5}x - \frac{3}{5} + \frac{1}{5}x^{-1}$, so $f'(x) = \frac{4}{5} - \frac{1}{5x^2}$ [or $\frac{4}{5} - \frac{1}{5}x^{-2}$]

12 $f'(x) = 8x + 2x^{-2}$

$f'(2) = 16 + \frac{2}{4} = 16.5$

13 $\frac{dy}{dx} = 12 - 5x^{-2} = 2$

$\frac{5}{x^2} = 10, x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

14 $\frac{dy}{dx} = 2x = 8, y = 16 - 3 = 13$

Tangent: $y - 13 = 8(x - 4)$

15 $\frac{dy}{dx} = 3 + 2x^{-2} = 3 + \frac{2}{4} = \frac{7}{2}$

$y = 6 - \frac{2}{2} = 5$

Normal: $y - 5 = -\frac{2}{7}(x - 2)$

$y = -\frac{2}{7}x + \frac{39}{7}$

16 $\frac{dy}{dx} = 2x$, so the tangent at $(a, a^2 - 3)$ is:

$y - (a^2 - 3) = 2a(x - a)$

When $x = 0, y = -12$:

$-9 - a^2 = -2a^2$

$a = \pm 3$

17 Use GDC to find the gradient and to draw the tangent.

-0.021

$y = -0.021x + 0.13$

18 Use GDC to sketch the graph of $\frac{dy}{dx}$ and intersect it with $y = 2$. The coordinates are $(0.5, -0.098)$.

19 $3x^3 - 3x^{-2} + c$

20 $\int \frac{1}{2}x^3 - \frac{3}{2}x^{-2}dx = \frac{1}{8}x^4 + \frac{3}{2}x^{-1} + c$

21 Integrate: $y = \int 4x + 2\, dx = 2x^2 + 2x + c$

Use $y = 3, x = 2$: $3 = 2(2^2) + 2(2) + c \Rightarrow c = -9$

So $y = 2x^2 + 2x - 9$

22 Use GDC: $\int_2^3 2x^3 - 1\, dx = 31.5$

23 $\frac{dy}{dx} = 3\cos x + 5\sin x$

24 $f'(x) = \frac{1}{\sqrt{x}} - \frac{3}{x}$

$f'(9) = \frac{1}{3} - \frac{3}{9} = 0$

25 a Using the chain rule with $y = u^{\frac{1}{2}}$ and $u = 3x^2 - 1$:

$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 1)^{-\frac{1}{2}}(6x) = \frac{3x}{\sqrt{3x^2 - 1}}$$

b Using the chain rule with $y = 2u^3$ and $u = \sin(5x)$, and remembering that $\sin(5x)$ differentiates to $5\cos(5x)$:

$$\frac{dy}{dx} = 6\sin^2(5x) \times 5\cos(5x) = 30\sin^2(5x)\cos(5x)$$

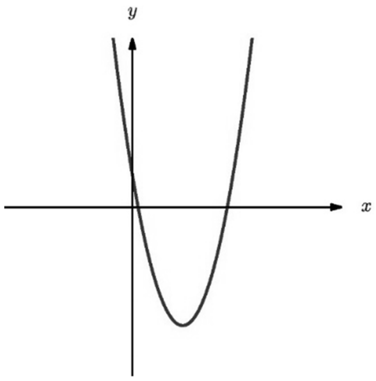
26 $4e^{-3x} - 12xe^{-3x}$

27 Using the quotient rule with $u = \ln x$ and $v = 4x$:

$$\frac{dy}{dx} = \frac{4x\frac{1}{x} - 4\ln x}{16x^2} = \frac{4 - 4\ln x}{16x^2} = \frac{1 - \ln x}{4x^2}$$

28 $6x + \frac{3}{x^2}$

29 The curve is concave-down in the middle section so $f''(x) < 0$ there.



30 Concave-down means that the graph curves downwards, which is at the points B, D and E.

31 Concave up means $f''(x) > 0$. So $f''(x) = 30x - 4 > 0 \therefore x > \frac{2}{15}$

32 $\frac{dy}{dx} = 3x^2 - \frac{24}{x} = 0 \Leftrightarrow 3x^3 = 24 \Leftrightarrow x = 2$

33 $f'(x) = \cos x + \sin x$, $f'\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

$f''(x) = -\sin x + \cos x$, $f''\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} < 0$

$y = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

34 $S = x^2 + 4 \times \left(x \times \frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

$\frac{dS}{dx} = 2x - \frac{128}{x^2} = 0$ when $x^3 = 64$ so $x = 4$.

$\frac{d^2S}{dx^2} = 2 + \frac{256}{x^3} > 0$, so this is a local minimum.

Min surface area $S = 4^2 + \frac{128}{4} = 48\ \text{cm}^2$

35 $\frac{d^2y}{dx^2} = 60x^3 - 120x^2 = 60x^2(x-2)$

So $\frac{d^2y}{dx^2} = 0$ when $x = 0$ or 2.

Check for change in sign of $\frac{d^2y}{dx^2}$:

x	-1	1	3
$\frac{d^2y}{dx^2}$	< 0	< 0	> 0

The only point of inflexion is at $x = 2$.

36 $v = \frac{ds}{dt} = 15\cos(5t)$, $a = \frac{dv}{dt} = -75\sin(5t)$

When $t = 2$, $a = 40.8\ \text{m s}^{-2}$

(or find $\frac{d^2s}{dt^2}$ at $t = 2$ using GDC)

37 Velocity: $v = \frac{ds}{dt} = -0.6e^{-0.2t} = -0.270$

(or find $\frac{ds}{dt}$ using GDC)

So speed = 0.270 m s^{-1}

38 $4 + \int_2^5 \frac{1}{\sqrt{t+3}}\,dt = 5.18$ m

39 13.8 m

40 $\left(2 \div \frac{1}{3}\right)x^{\frac{1}{3}} + \frac{4}{3}\ln|x| = 6x^{\frac{1}{3}} + \frac{4}{3}\ln|x| + c$

41 $\frac{2e^{4x}}{4} + \frac{3e^{-\frac{1}{3}x}}{-\frac{1}{3}} = \frac{1}{2}e^{4x} - 9e^{-\frac{1}{3}x} + c$

42 a $4 \times \frac{\sin^3 x}{3} = \frac{4}{3}\sin^3 x + c$

b $\frac{1}{2}\int\left(\frac{2x}{x^2+3}\right)dx = \frac{1}{2}\ln(x^2+3) + c$

43 $\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{6}} = -\frac{1}{2}\left(\frac{1}{2} - 1\right) = \frac{1}{4}$

44 1.79 (3 s.f.)

45 $\int_0^{\frac{\pi}{2}} (\sin x - 1)\, dx = [-\cos x - x]_0^{\frac{\pi}{2}}$

$\left[0 - \frac{\pi}{2}\right] - [-1 - 0] = 1 - \frac{\pi}{2}$

So area $= \frac{\pi}{2} - 1$.

46 $e^{-x} = \frac{1}{3}, e^{x} = 3;\ \ x = \ln 3$. The coordinates are (ln 3, 0)

$$\int_0^{\ln 3} (3e^{-x} - 1)\, dx = [-3e^{-x} - x]_0^{\ln 3} = \left[-\frac{3}{e^{\ln 3}} - \ln 3\right] - [-3e^0 - 0] = -\frac{3}{3} - \ln 3 + 3$$
$$= 2 - \ln 3$$

$$\int_{\ln 3}^{2} (3e^{-x} - 1)\, dx = [-3e^{-x} - x]_{\ln 3}^{2} = [-3e^{-2} - 2] - \left[-\frac{3}{3} - \ln 3\right] = -3e^{-2} - 1 + \ln 3$$

Area = $2 - \ln 3\ + 3e^{-2} + 1 - \ln 3 = 3 - 2\ln 3 + 3e^{-2}$

47 a Using GDC, points of intersection are $(-4.82, 0.180), (2.69, 7.69)$

b Subtract the bottom curve from the top curve and integrate:

$$A = \int_{-4.82}^{2.69} x + 5 - 2e^{0.5x}\ \ dx = 14.6$$